When Everyone Upgrades:

Replacement Complementarities and Endogenous Growth

Nicolò Ceneri*

New York University

October 2026

Abstract

Households replace durable goods in response to peers upgrading. Replacement purchases are the market for new products, but, at the same time, innovation makes existing goods obsolete sooner. This loop gives rise to multiplicity: in one equilibrium, replacement is rare and innovation is slow; in the other, replacement is frequent and innovation is fast. Multiplicity requires that durable services fall with the distance between the quality of a buyer's durable and the average quality of peers' durables. Calibrated to the peer response in phone purchases, a cost increase that cuts the seller's margin by more than half a percent removes the fast equilibrium and lowers growth from 3 to about 1.3 percent a year.

Keywords: durable replacement; peer effects; endogenous growth; product innovation; replacement complementarities; equilibrium multiplicity.

JEL Codes: O31, O41, D12, D90.

*Department of Economics, New York University, USA. Corresponding author: nicolo.ceneri@nyu.edu.
I am indebted to Guido Menzio and Jess Benhabib for their guidance, and thank Paolo Martellini and participants of the NYU Macro Lunch for their comments.

1 Introduction

Households replace phones, cars, and computers while the old units still provide services. In standard models of durable adjustment, the timing of purchases depends on the gain from adjusting the durable stock and on the transaction cost of adjustment (Grossman and Laroque, 1990; Eberly, 1994). I add a complementarity: durable services fall with the distance between the quality of a household's durable and the average quality of peers' durables. Since the return to innovation depends on the frequency of replacement, this complementarity can give rise to multiple equilibria. In one, households replace goods infrequently and innovation is slow; in the other, replacement is frequent and innovation is fast.

Evidence on phone purchases supports the peer effect. Bailey et al. (2022) exploit random phone losses among friends and find that a friend's new phone purchase raises demand for phones of the same brand. In the model, the peer effect acts on replacement timing.

How do peer effects shape durable replacement, and how does replacement feed back into product innovation and long-run growth? Peer upgrading raises a buyer's gain from replacement. More replacement purchases expand the seller's market. A larger market raises frontier growth, and faster frontier growth makes installed durables obsolete sooner. Durable replacement and product innovation are therefore complements.

In the model, each buyer owns one durable good, sold in differentiated product sublines. Each product subline has an innovating seller. Each buyer chooses when to replace its durable with the frontier quality. Durable services depend on the quality of the buyer's vintage and on its distance from a peer benchmark. The peer benchmark combines the vintages other buyers own and the quality at which they replace. Sellers choose prices and frontier growth.

Multiplicity arises: a slow-replacement equilibrium and a fast-replacement equilibrium. In the slow-replacement equilibrium, peers hold older vintages, buyers keep their durables until they fall far below frontier quality, and frontier growth is slow. In the fast-replacement equilibrium, peers hold newer vintages, buyers replace close to frontier quality, and frontier growth is rapid. A middle equilibrium separates the two. Physical depreciation is the same in the three equilibria. Faster frontier growth makes installed durables obsolete sooner, so replacement frequencies differ.

Multiplicity relies on the distance between frontier quality and the average quality of peers' durables. When replacement is slow, a new durable is far from that average, and this distance makes buyers wait longer.

The model predicts that an improvement in durables held by peers raises replacement incentives most for buyers holding the oldest vintages.

Figure 1 shows cross-country patterns in durable replacement, proxied by disposal.

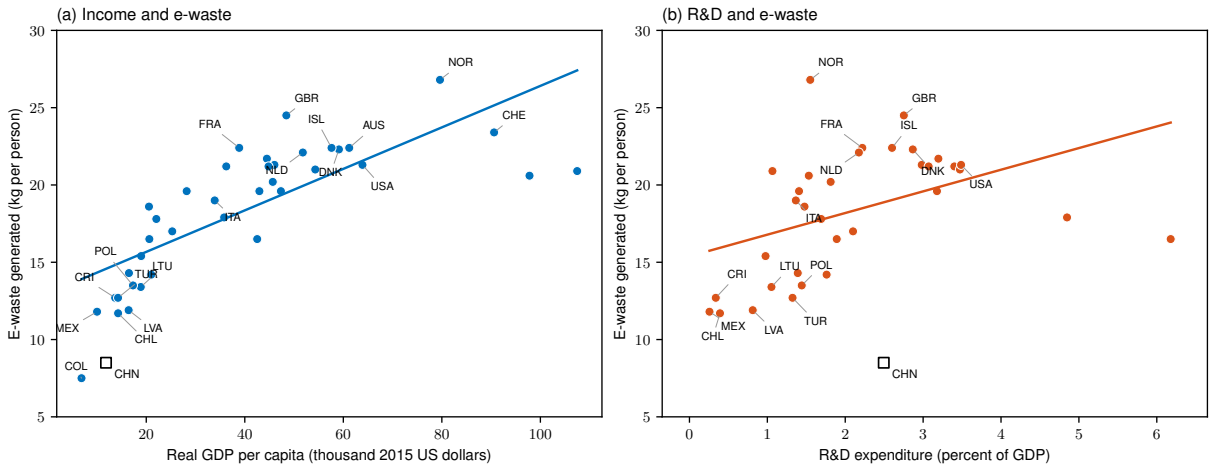


Figure 1: Electronic waste, income, and research. Panels (a) and (b) use 2022 OECD countries: generated e-waste against real GDP per capita (38 countries) and against R&D expenditure as a share of GDP (34 countries). Lines are separate univariate least-squares fits. Country labels mark the eight OECD countries with the highest e-waste, the eight with the lowest, Italy, and the United States. China is shown as an open square; the lines fit the OECD countries. Generated e-waste includes discarded electrical and electronic equipment before collection or recycling. Replacement purchases are a distinct model object. Sources: Baldé et al. (2024), Annex 2, Table A2.4; and World Development Indicators, series NY.GDP.PCAP.KD and GB.XPD.RSDV.GD.ZS. Construction is given in Appendix A.4.

OECD countries with higher real GDP per person generate more electronic waste per person, and electronic waste rises with R&D expenditure. In the model, replacement and seller R&D intensity rise together across stationary equilibria. The quantification maps electronic waste to replacement through a common product mass.

I calibrate the model to the peer response in phone purchases, the relative price decline of durable goods, vehicle turnover, and car depreciation. Buyers replace every 14.7 years in the slow-replacement equilibrium and every 7.3 years in the fast-replacement equilibrium, and frontier growth is 1.31 and 2.99 percent per year.

I use the model to study a current increase in the production cost of durable goods. Demand for memory from AI data centers has raised component costs for PC and smartphone producers (TrendForce, 2026), and industry forecasts project lower device shipments and longer device lifetimes (Gartner, 2026). In the model, production cost rises and the gross markup falls, so the seller margin falls and the buyer price rises. In the slow- and fast-replacement equilibria, buyers keep their durables longer, the seller's market shrinks, and frontier growth falls. A seller-margin loss of 0.5 percent lowers frontier growth in the fast-replacement equilibrium from 2.99 to 2.74 percent per year. A seller-margin loss above 0.55 percent removes the fast-replacement equilibrium. At a 10 percent loss, the replacement age rises from 7.3 to 16.6 years, frontier growth falls to 1.03 percent per year, and after 30 years frontier quality is 45 percent below its initial path. A temporary cost increase leaves

a permanent gap in frontier quality.

Related literature. The paper relates to work on durable replacement, endogenous growth, and adoption with peers.

Durable replacement. Transaction costs generate infrequent purchases in Grossman and Laroque (1990). Caballero (1993) connects lumpy household purchases to the slow adjustment of aggregate durable spending. Caplin and Leahy (2006) derive equilibrium durable demand from the distribution of holdings when purchases are lumpy. Eberly (1994) estimates a replacement rule for automobiles and links the distribution of household durable holdings to aggregate purchases. Berger and Vavra (2015) show how fixed adjustment costs and that distribution shape the response of durable spending to shocks. Bertolotti et al. (2023) link new-car quality to durable expenditures, and Goettler and Gordon (2011) jointly model upgrades and innovation in the microprocessor market. In this paper, peer upgrading changes buyers' willingness to wait, replacement responds to the installed vintages of other buyers, and replacement sales determine the market for the seller's next vintage.

Endogenous growth. Embodied technical change makes the quality of new goods a source of growth (Greenwood et al., 1997). Grossman and Helpman (1991) model endogenous quality improvements along product lines. Perla and Tonetti (2014) connect technology upgrading to the distribution of technologies in use. Benhabib et al. (2021) jointly determine innovation, adoption, and that distribution. Aghion et al. (2025) direct innovation toward material productivity or final-use quality and study the environmental implications of quality-led growth. Bertolotti and Lanteri (2024) study a replacement–innovation feedback for capital goods: capital replacement creates sales for producers of new capital, and innovation raises the quality available to firms that replace. On the supply side, Bertolotti et al. (2024) study how investment-goods producers' market power affects capital accumulation. In this paper, durable replacement generates the market for product innovation, and the peer benchmark enters each buyer's replacement threshold. Peer comparisons sustain different replacement flows and frontier growth rates under the same fundamentals.

Adoption with peers. The installed base affects technology adoption in Katz and Shapiro (1985); Farrell and Saloner (1985). Alvarez et al. (2023a) study adoption of digital payments when the benefit of adoption rises with adoption by others; their model generates gradual adoption and multiple equilibria. Ceneri (2026b) studies network coordination in technology adoption and growth. Individual choices depend on a distribution of other agents' states in mean-field games (Lasry and Lions, 2007), in the dynamic games with strategic complementarities of Adlakha and Johari (2013), and in the price-setting model of Alvarez et al. (2023b). In this paper, buyers replace durables repeatedly as quality improves, and the

distribution of vintages in use determines the peer benchmark and the seller's market.

2 Environment

Time is continuous. A unit mass of buyers each owns one durable good, such as a phone. The good is sold in a unit interval of differentiated product sublines indexed by $j \in [0,1]$. Each product subline has a seller with exclusive access to its frontier vintage and offers that vintage alone; Section 4.1 reports the buyer's gain from older vintages. Buyers choose when to replace their durables and which product subline to purchase. Sellers choose prices and frontier growth. All agents discount at rate $\rho > 0$.

Buyers and ownership. Buyers receive durable services and consume the numeraire. Buyers own equal diversified claims to the sellers, and sellers' profits return as dividends. Each buyer takes the peer benchmark, the frontier path, and the dividend as given. Buyers collectively determine replacement sales and the peer benchmark.

Product sublines and production. Let $\bar{x}_j(t)$ be the frontier quality of product subline j. Seller j uses $n_j(t)$ units of the numeraire input to produce $y_j(t)$ units of its frontier durable according to

$$y_j(t) = \frac{n_j(t)}{\kappa \bar{x}_j(t)}, \qquad \kappa > 0. \tag{1}$$

Marginal cost is $\kappa \bar{x}_j(t)$. Higher frontier quality requires more input per unit.

Let $\mathcal{J}(t)$ be the mass of buyers that replace per unit time. Conditional on replacement, buyers generate CES demand across product sublines with elasticity $\sigma > 1$. Let $p_j(t)$ be seller j's price and let $P(t)$ be the quality-adjusted price index,

$$P(t) = \left[\int_0^1 \left(\frac{p_j(t)}{\bar{x}_j(t)} \right)^{1-\sigma} \mathrm{d}j \right]^{\frac{1}{1-\sigma}}. \tag{2}$$

Purchases from product subline j are

$$d_j(t) = \mathcal{J}(t) \left[\frac{p_j(t)/\bar{x}_j(t)}{P(t)} \right]^{-\sigma}. \tag{3}$$

Seller j takes $\mathcal{J}(t)$ and $P(t)$ as given and solves

$$\max_{p_j(t)} \Big[p_j(t) - \kappa \bar{x}_j(t) \Big] d_j(t). \tag{4}$$

The price first-order condition gives

$$p_j(t) = \eta\kappa\bar{x}_j(t), \qquad \eta \equiv \frac{\sigma}{\sigma - 1} > 1. \tag{5}$$

In a symmetric equilibrium, $\bar{x}_j(t) = \bar{x}(t)$, $p_j(t) = p(t)$, and $d_j(t) = \mathcal{J}(t)$ for every product subline. Define the seller margin per unit of frontier quality as

$$\pi^S \equiv (\eta - 1)\kappa. \tag{6}$$

The symmetric frontier price is

$$p(\bar{x}(t), t) = \eta\kappa\bar{x}(t) = (\kappa + \pi^S)\bar{x}(t). \tag{7}$$

Durable quality. Let $x_i(t)$ be the effective service quality of buyer i's installed durable. Between replacements, installed quality depreciates at rate $\delta \geq 0$:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\delta x_i(t). \tag{8}$$

The common frontier $\bar{x}(t)$ grows at the rate $\gamma(t)$ chosen by each seller,

$$\frac{\dot{\bar{x}}(t)}{\bar{x}(t)} = \gamma(t) \geq 0. \tag{9}$$

Frontier growth $\gamma(t)$ costs $\omega\gamma(t)^2\bar{x}(t)$ per unit time, with $\omega > 0$.

Let $\underline{x}(t)$ be the replacement threshold and the lower endpoint of the in-use support. Let $f(x, t)$ be the density of installed qualities on $[\underline{x}(t), \bar{x}(t)]$. Average installed quality is

$$X(t) = \int_{\underline{x}(t)}^{\bar{x}(t)} x f(x, t)\,\mathrm{d}x. \tag{10}$$

Peer comparisons and durable services. The peer benchmark combines the replacement threshold and average installed quality:

$$M(t) = (1 - \zeta)\log\underline{x}(t) + \zeta\log X(t), \qquad \zeta \in [0, 1]. \tag{11}$$

The threshold is the quality at which peers discard their old durables. Average installed quality is the quality they hold. The peer weight ζ sets the relative weight on these two statistics.

Let $\mathcal{A}(t) > 0$ be service value per unit of quality and let $\mathcal{B}(t) \geq 0$ be mismatch curvature.

Buyer i receives the flow service payoff

$$U(x_i, t) = \mathcal{A}(t)x_i(t) - \mathcal{B}(t)[\log x_i(t) - M(t)]^2. \tag{12}$$

Let $c_i(t)$ be buyer i's numeraire consumption. Numeraire consumption enters utility linearly. The buyer's lifetime utility is

$$\int_0^\infty e^{-\rho t}\Big[c_i(t) + U(x_i, t)\Big]\,\mathrm{d}t. \tag{13}$$

The first term values the durable's own quality. The second term lowers services with the distance between installed quality and peers' quality, summarized by $M(t)$.

Replacement. Each buyer observes its installed quality, the frontier, and the peer benchmark. Replacement discards the installed durable and resets quality to the frontier. A replacement costs the frontier price plus the fixed replacement cost $\chi\underline{x}(t)$. The fixed replacement cost scales with the aggregate replacement threshold, which each buyer takes as given. Threshold scaling keeps the fixed replacement cost constant relative to installed qualities along a growing path.

Let $V(x, t)$ be the buyer's value. In the continuation region, V satisfies the Hamilton–Jacobi–Bellman (HJB) equation

$$\rho V(x, t) = U(x, t) + V_t(x, t) - \delta x V_x(x, t). \tag{14}$$

Value matching and smooth pasting at the replacement threshold give

$$V(\underline{x}(t), t) = V(\bar{x}(t), t) - p(\bar{x}(t), t) - \chi\underline{x}(t), \tag{15}$$

$$V_x(\underline{x}(t), t) = 0. \tag{16}$$

The buyer takes $M(t)$, $\underline{x}(t)$, and $\bar{x}(t)$ as given. The replacement threshold is optimal when these conditions hold.

Replacement also determines the distribution of installed qualities. Let $\boldsymbol{\delta}(\cdot)$ denote the Dirac function. The density follows the Kolmogorov forward equation (KFE)

$$f_t + \partial_x[-\delta x f] = \mathcal{J}(t)\boldsymbol{\delta}(x - \bar{x}(t)) - \mathcal{J}(t)\boldsymbol{\delta}(x - \underline{x}(t)). \tag{17}$$

Mass reaching the threshold leaves the installed stock and re-enters at the frontier.

Innovating sellers. The markup in (5) gives the seller profit $\pi^S \bar{x}(t)$ on each replacement sale. Flow profit is $\pi^S \mathcal{J}(t)\bar{x}(t)$. Each seller takes $\mathcal{J}(t)$, the replacement threshold, and the peer benchmark as given. Let $S(\bar{x}, t)$ be the seller's value. The seller solves

$$\rho S(\bar{x}, t) = \max_{\gamma \geq 0} \Big\{ \pi^S \mathcal{J}(t)\bar{x} - \omega\gamma^2 \bar{x} + S_{\bar{x}}(\bar{x}, t)\gamma\bar{x} + S_t(\bar{x}, t) \Big\}. \tag{18}$$

Write $S(\bar{x}, t) = s(t)\bar{x}(t)$. The innovation first-order condition is

$$\gamma(t) = \frac{s(t)}{2\omega}. \tag{19}$$

A larger replacement flow raises the value of frontier quality and the chosen frontier growth.

Definition 1. *A symmetric recursive equilibrium consists of buyer and seller values, prices, production, replacement and innovation policies, a peer benchmark, and an installed-quality density such that, at every aggregate state:*

(i) buyers solve (14) subject to (15)–(16);

(ii) sellers solve the price problem (4) and the innovation problem (18);

(iii) product-subline demand satisfies (3), production satisfies (1), and replacement sales equal $\mathcal{J}(t)$ in symmetry; and

(iv) the density satisfies (17), average quality satisfies (10), and the peer benchmark satisfies (11).

3 Balanced Growth Equilibrium

The frontier, the replacement threshold, and all installed qualities grow with the economy. Dividing by the replacement threshold gives constant normalized quantities along a balanced-growth path.

3.1 Normalization

Define the threshold growth rate, the replacement range, the normalized quality index, and the effective depreciation drift:

$$g(t) \equiv \frac{\mathrm{d} \log \underline{x}(t)}{\mathrm{d}t}, \qquad q^*(t) \equiv \log\left(\frac{\bar{x}(t)}{\underline{x}(t)}\right),$$

$$q \equiv \log\left(\frac{x}{\underline{x}(t)}\right) \in [0, q^*(t)], \qquad \Delta(t) \equiv \delta + g(t).$$

In these units, $q = 0$ is the replacement threshold and $q = q^*(t)$ is the frontier. An unreplaced vintage drifts toward $q = 0$ at speed $\Delta(t)$, the sum of the physical depreciation rate and threshold growth. At $q = 0$, the buyer replaces it and resets to q^*.

Let $\psi(q, t)$ denote the density of installed qualities in normalized units, defined by the change of variables $\psi(q, t)\, \mathrm{d}q = f(x, t)\, \mathrm{d}x$ from the levels density f in (17). Threshold-normalized average installed quality is

$$Q(t) \equiv \log\left(\int_0^{q^*(t)} e^q\, \psi(q, t)\, \mathrm{d}q\right) = \log\left(\frac{X(t)}{\underline{x}(t)}\right),$$

and the peer benchmark (11) in normalized units becomes $M(t) - \log \underline{x}(t) = \zeta\, Q(t)$.

Normalized payoffs are stationary when $\mathcal{A}(t) = A$ and $\mathcal{B}(t) = B\, \underline{x}(t)$ for constants $A > 0$ and $B \geq 0$.

Dividing the service payoff (12) by $\underline{x}(t)$ and subtracting the state-independent threshold-service constant gives the normalized flow payoff used in the buyer's stopping problem:

$$u(q, Q) = A\,(e^q - 1) - B(q - \zeta Q)^2. \tag{20}$$

3.2 Balanced growth paths

Definition 2. *A BGP is an equilibrium in which the frontier and the replacement threshold grow at a common constant rate $g \in (0, \rho)$, and the buyer value v, the installed-quality density ψ, the seller shadow value s, the replacement range q^*, and the normalized average quality Q are time-invariant.*

The restriction $g < \rho$ keeps discounted normalized values finite. On a BGP, $\Delta = \delta + g$ is constant, and the normalized buyer value solves

$$(\rho - g)\, v(q) = A(e^q - 1) - B(q - \zeta Q)^2 - \Delta\, v'(q), \qquad q \in (0, q^*), \tag{21}$$

with stopping conditions

$$v(q^*) - v(0) = (\kappa + \pi^S)\, e^{q^*} + \chi, \qquad v'(0) = 0. \tag{22}$$

Between replacements, each vintage drifts deterministically toward $q = 0$ at constant speed Δ. At $q = 0$, the buyer replaces it, and the vintage returns to $q = q^*$. Each vintage completes a replacement cycle in q^*/Δ years. The KFE is a transport equation and preserves the shape of the initial distribution.

Proposition 1. *Suppose that the initial distribution is absolutely continuous, then any BGP has uniform normalized density:*

$$\psi(q) = \frac{1}{q^*}, \qquad q \in [0, q^*].$$

The replacement flow per unit buyer mass and the normalized average quality are

$$\mathcal{J}(q^*, g) = \frac{\delta + g}{q^*}, \qquad Q(q^*) = \log\left(\frac{e^{q^*} - 1}{q^*}\right).$$

An unreplaced vintage moves toward the replacement threshold at speed $\delta + g$ and crosses a distance q^*. A wider replacement range means buyers replace less often and the market for new durables is thinner. The replacement flow $\mathcal{J}$ is the demand that the seller faces when choosing frontier growth.

Let $r = x/\bar{x} = e^{q-q^*}$ be owned durable quality relative to the frontier. A buyer replaces when r reaches e^{-q^*}, and newly purchased durables have $r = 1$. Under the stationary distribution in Proposition 1, the density of r is $f_r(r) = 1/(q^* r)$ on $[e^{-q^*}, 1]$. Uniform mass over q produces a density that falls in r. Figure 2 plots this density and its cumulative distribution at the slow-replacement equilibrium L and the fast-replacement equilibrium H of the reference economy in Section 5. Buyers replace at $r \approx 0.13$ in L and at $r \approx 0.32$ in H. Durables with $r \geq 0.8$ account for 11 percent of buyers in L and 20 percent in H.

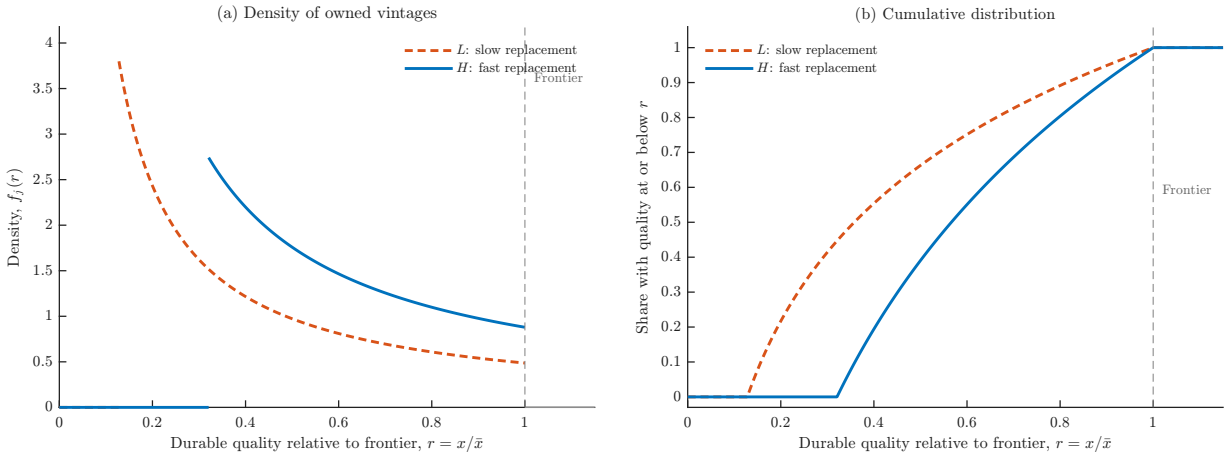


Figure 2: Stationary distribution of buyers' owned durable quality relative to the frontier. The horizontal axis is $r = x/\bar{x}$; the dashed line at $r = 1$ marks frontier-quality durables. Panel (a) shows the density and panel (b) the cumulative share. L and H are the slow- and fast-replacement stationary equilibria at the reference parameter values in Table 1.

The normalized average quality can equivalently be decomposed as

$$Q(q^*) = \underbrace{\frac{q^*}{2}}_{\substack{\text{midpoint of the} \\ \text{uniform support}}} + \underbrace{\log\left(\frac{2\sinh(q^*/2)}{q^*}\right)}_{\text{Jensen correction}}. \tag{23}$$

The first term is the mean of the uniform density on $[0, q^*]$. The second term is the quality gap. Q is the log of the average level $X/\underline{x} = \int e^q \psi \, \mathrm{d}q$, so by Jensen's inequality Q exceeds the average log quality $q^*/2$. The gap is small when the replacement range is narrow and rises with q^*.

The derivative of Q with respect to the replacement range is

$$Q'(q^*) = \underbrace{\frac{1}{2}}_{\substack{\text{midpoint} \\ \text{shift}}} + \underbrace{\frac{1}{2}\coth(q^*/2) - \frac{1}{q^*}}_{\substack{\text{quality gap} \\ \text{response to wider support}}} > 0. \tag{24}$$

A wider replacement range raises average installed quality relative to the replacement threshold and lowers it relative to the frontier. A higher Q raises the peer benchmark ζQ, which raises the mismatch loss for buyers below the benchmark and lowers it for buyers above it. The variable q is measured from the quality at which peers discard their durables. At $\zeta = 0$, the mismatch loss Bq^2 therefore depends on the replacement threshold alone.

4 Stationary Equilibrium and Multiplicity

Definition 3. *A BGP root (q^*, g) is* regular *if: $q^* \in (0, \infty)$ and $g \in (0, \rho)$; value matching and smooth pasting hold; the stopping boundary is strict, $v(q) > v(0)$ for all $q \in (0, q^*]$; the seller's innovation choice is interior, $s = 2\omega g > 0$; and the Jacobian of the stationary buyer–seller system is nonsingular.*

The buyer and seller conditions jointly determine the replacement range q^* and the frontier-growth rate g. Let $\Theta = (A, B, \rho, \delta, \omega, \pi^S, \kappa, \chi)$ collect the stationary coefficients apart from the peer weight ζ. The pricing relation (6) links π^S to η and κ.

Taking average installed quality Q as given, the buyer solves a single-agent stopping problem. Solving the buyer equation (21) with smooth pasting and imposing the consistency condition $Q = Q(q^*)$ from Proposition 1 defines the buyer indifference residual:

$$F_B(q^*, g; \Theta, \zeta) \equiv \underbrace{v(q^*) - v(0)}_{\substack{\text{value gain from} \\ \text{resetting to frontier}}} - \underbrace{(\kappa + \pi^S)\, e^{q^*}}_{\substack{\text{normalized} \\ \text{frontier price}}} - \underbrace{\chi}_{\substack{\text{fixed} \\ \text{replacement cost}}}. \tag{25}$$

The residual is the net benefit of replacement at the candidate range q^*. When $F_B > 0$, the gain exceeds the cost: the gap is too wide and buyers want to replace sooner. When $F_B < 0$, the cost exceeds the gain and buyers prefer to wait longer. At a BGP, $F_B = 0$ and the buyer is indifferent at the threshold. Integrating the buyer differential equation gives the residual in Appendix B.3.

The seller's HJB and first-order condition from Section 2 collapse on a BGP to a single condition relating the replacement range to the growth rate:

$$F_S(q^*, g; \Theta) \equiv \underbrace{\pi^S \frac{\delta + g}{q^*}}_{\substack{\text{flow profit from} \\ \text{frontier sales}}} - \underbrace{\omega\, g\,(2\rho - g)}_{\substack{\text{effective cost of} \\ \text{sustaining growth}}} = 0. \tag{26}$$

The first term is the seller's profit: each frontier sale earns the margin π^S, and sales arrive at the replacement flow $(\delta + g)/q^*$ from Proposition 1. The second term is the full cost of sustaining frontier growth. It equals the direct flow R&D cost ωg^2 plus the required return $(\rho - g)s = 2\omega(\rho - g)g$ on the shadow value of frontier quality. A larger replacement flow supports faster frontier growth.

Equation (26) inverts to the seller curve:

$$q_S(g; \Theta) = \frac{\overbrace{\pi^S(\delta + g)}^{\text{seller's gross market}}}{\underbrace{\omega\, g\,(2\rho - g)}_{\text{effective innovation cost}}}. \tag{27}$$

For each growth rate, the seller curve gives the replacement range at which frontier sales make that growth rate optimal for the seller. A higher margin π^S and a faster drift $\delta + g$ toward the replacement threshold raise the return to frontier innovation. The denominator is the cost of sustaining growth. As g approaches zero, the denominator approaches zero and q_S diverges: slow growth is consistent with a very wide replacement range and a very small replacement flow. Near ρ, the denominator flattens while the numerator rises with g, so the seller curve has an interior minimum on $(0, \rho)$.

Proposition 2. *Let $\beta \equiv (\rho - g)/(\delta + g)$. For any $q^* > 0$ and $g \in (0, \rho)$,*

$$\Xi_1(q^*, \Delta, \beta) = \frac{2}{\Delta}\left[\frac{q^*}{\beta} - \frac{1 - e^{-\beta q^*}}{\beta^2}\right] > 0. \tag{28}$$

The response of the buyer residual to average peer quality is

$$\frac{\partial F_B}{\partial Q} = B\zeta\, \Xi_1(q^*, \Delta, \beta).$$

Hence $\partial F_B/\partial Q > 0$ *if and only if* $B\zeta > 0$.

A higher aggregate quality makes buyers more willing to replace their old durables, increasing sales and the seller's return to innovation. The strength of this average-quality channel is proportional to $B\zeta$. But the replacement threshold also affects buyers' decisions at every ζ, since quality q is measured relative to the quality at which peers replace their durables. At $\zeta = 0$, the peer benchmark equals the replacement threshold, and the term Bq^2 lowers services with the distance from the replacement threshold. For a narrow range of B, this effect alone is enough to generate three stationary equilibria (Figure 5).

Substituting the seller curve (27) into the buyer residual gives one equation in the frontier-growth rate:

$$\Psi(g; \Theta, \zeta) \equiv F_B\Big(q_S(g; \Theta),\, g;\, \Theta,\, \zeta\Big) = 0 \tag{29}$$

A zero of Ψ satisfies the seller condition and the buyer condition.

The buyer residual is affine in the peer weight (Appendix B.3),

$$\Psi(g; \Theta, \zeta) = \underbrace{S_0(g; \Theta)}_{\substack{\text{buyer residual} \\ \text{at } \zeta=0}} + \zeta \quad \underbrace{S_1(g; \Theta)}_{\substack{\text{marginal effect} \\ \text{of the peer weight}}}, \qquad S_1(g; \Theta) > 0 \text{ for } B > 0. \tag{30}$$

The term S_0 is the buyer residual when the peer benchmark equals the replacement threshold. The term ζS_1 is the replacement incentive from the average-quality channel. Since $S_1 > 0$, a higher peer weight raises the buyer residual at every growth rate.

At the replacement range on the seller curve, buyer optimality holds when the peer weight equals the required peer weight

$$Z(g; \Theta) \equiv -\frac{S_0(g; \Theta)}{S_1(g; \Theta)}. \tag{31}$$

When the required peer weight is monotone in g, each peer weight corresponds to at most one stationary growth rate. When it changes direction, one peer weight corresponds to several growth rates, each with a different replacement flow.

Proposition 3. *Suppose* $B > 0$ *and let* $K \subset (0, \rho)$ *be a compact interval of growth rates. The set of regular BGPs on* K *at peer weight* ζ *is*

$$\mathcal{E}_K(\zeta) = \Big\{(q_S(g; \Theta),\, g) :\ g \in K,\ Z(g; \Theta) = \zeta,\ (q_S(g), g) \text{ regular}\Big\}.$$

Corollary 1. *If $B > 0$ and $Z(\cdot;\Theta)$ is strictly monotone on K, there is at most one regular BGP for each ζ. Conversely, if $B = 0$, the buyer residual is independent of ζ and $\mathcal{E}_K(\zeta)$ is constant in ζ.*

The $B = 0$ economy has the seller-side replacement–innovation feedback of Bertolotti and Lanteri (2024), in which replacement and embodied innovation are jointly determined. The mismatch payoff adds complementarity among buyers.

4.1 Multiplicity of stationary equilibria

A wider replacement range lowers current frontier sales. It also moves the quality at which peers discard their durables further below the frontier and, when $\zeta > 0$, raises threshold-normalized average quality. Both change the peer benchmark faced by buyers. Within the band of mismatch curvatures in Figure 5, this second force reverses the slope of the required peer weight.

Assumption 1 (Regular turning point)**.** *At a peer weight $\zeta_F \in (0,1)$, $\Psi(\cdot;\Theta,\zeta_F)$ has two zeros on K: a turning point g_F with $Z'(g_F;\Theta) = 0$ and $Z''(g_F;\Theta) \neq 0$, and a background root g_B with $\Psi_g(g_B;\Theta,\zeta_F) \neq 0$. Both roots are regular BGPs, and $|\Psi(\cdot;\Theta,\zeta_F)|$ is bounded away from zero on K outside neighborhoods of g_F and g_B.*

Theorem 1. *Suppose $B > 0$ and Assumption 1 holds. For peer weights sufficiently close to ζ_F, the same parameter vector supports three regular stationary equilibria when $(\zeta - \zeta_F)/Z''(g_F;\Theta) > 0$ and a unique regular stationary equilibrium when $(\zeta - \zeta_F)/Z''(g_F;\Theta) < 0$.*

In the slow-replacement equilibrium L, buyers keep durables longer and the seller chooses slower frontier growth, which sustains longer replacement ages. In the fast-replacement equilibrium H, buyers replace sooner and the seller chooses faster frontier growth, which lowers older durables relative to the peer benchmark. The middle equilibrium M has an intermediate replacement flow and frontier-growth rate.

Figure 3 shows the buyer's value and replacement policy in L and H. The panels (a) and (b) plot the gain from keeping an installed durable, $v(q) - v(0)$, against installed quality relative to the frontier. The gain equals zero at the replacement threshold and is positive in the continuation region. In both equilibria, the gain rises up to the frontier. The panels (c) and (d) show the replacement threshold in each equilibrium. Buyers retain durables over a wider quality range in L and replace them closer to the frontier in H.

Sellers offer the frontier vintage alone. Consider a buyer at the replacement threshold who can buy a durable of quality $r\bar{x}$, $r \leq 1$, at the same price per unit of quality, $(\kappa + \pi^S)r\bar{x}$. In each stationary equilibrium, the net gain is largest at a quality below the frontier. At a

price equal to marginal cost, $\kappa r \bar{x}$, the buyer also prefers a quality below the frontier. The restriction to the frontier vintage is therefore binding in each stationary equilibrium. A market for older vintages changes the replacement problem.

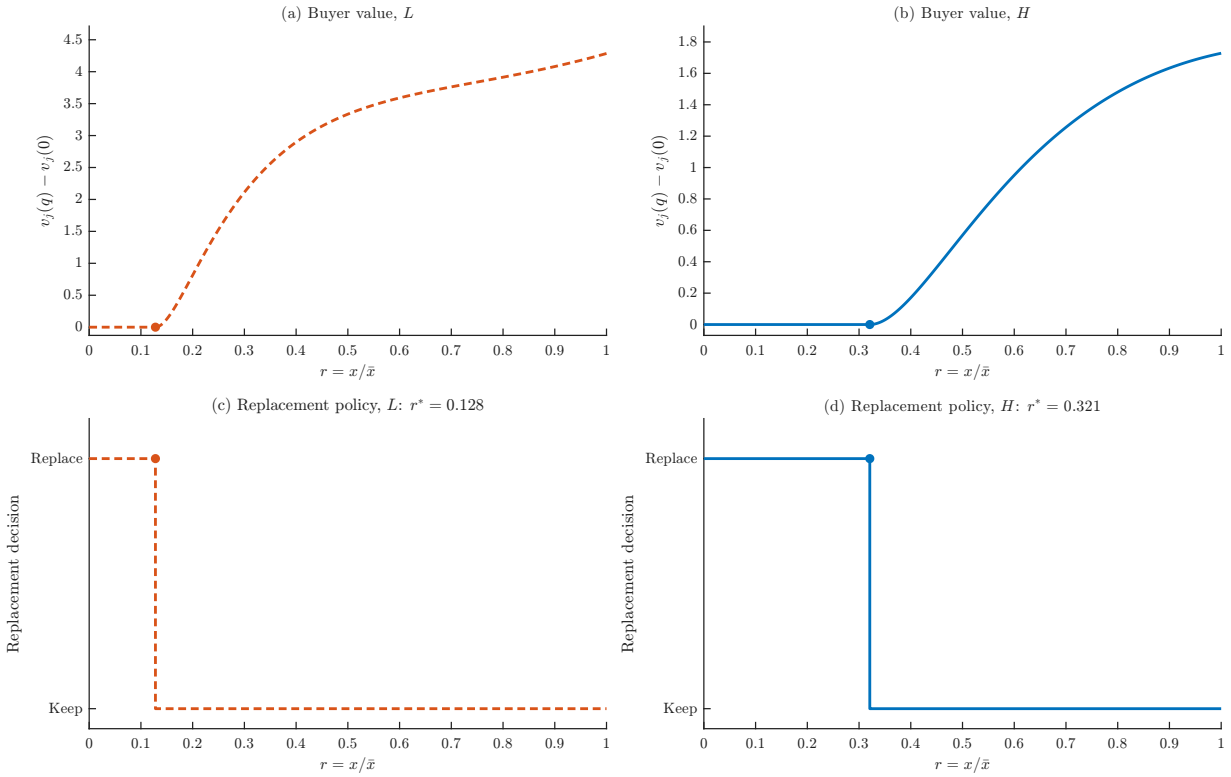


Figure 3: Buyer value relative to immediate replacement and optimal replacement policy in the slow (L) and fast (H) equilibria at the reference parameter values in Table 1. The value panels use different vertical scales.

The seller's value is proportional to frontier quality, $S(\bar{x}) = s\bar{x}$. Its innovation policy is $\gamma(s) = \max\{0, s/(2\omega)\}$. Figure 4 plots this policy and marks L and H. The larger replacement flow in H raises the shadow value of frontier quality and the chosen frontier-growth rate.

Figure 5 shows a region of peer weights and mismatch curvatures that supports several regular stationary equilibria, holding the other calibrated parameters fixed. Coexistence also occurs when the peer benchmark depends entirely on the replacement threshold. Peer replacement decisions therefore support coexistence through the replacement-threshold channel. As the peer weight rises, the coexistence region moves toward higher mismatch curvature. Lower and higher mismatch curvatures outside this region support fewer regular stationary equilibria.

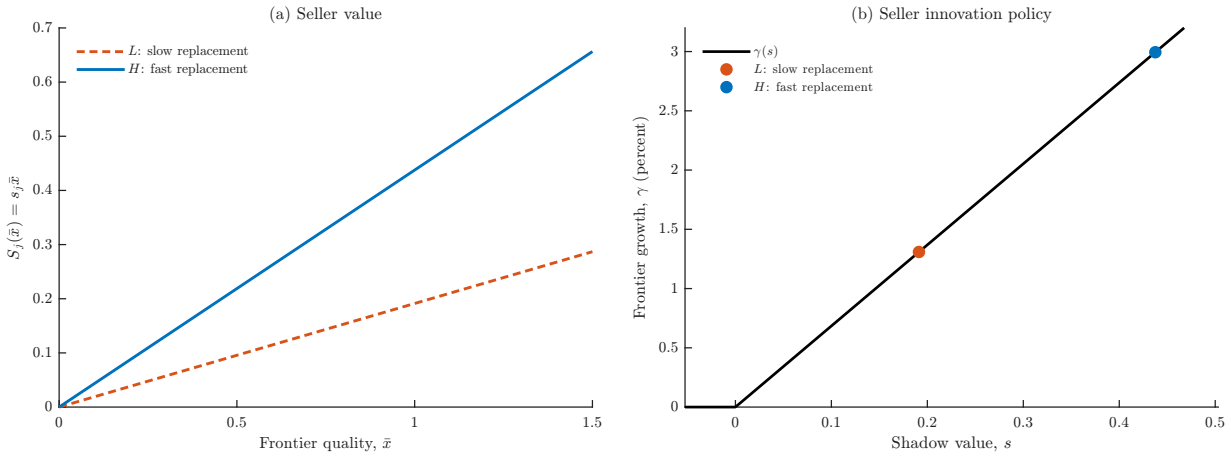


Figure 4: Innovating seller value and innovation policy. Markers in panel (b) are the slow- and fast-replacement stationary choices under the reference parameter values in Table 1.

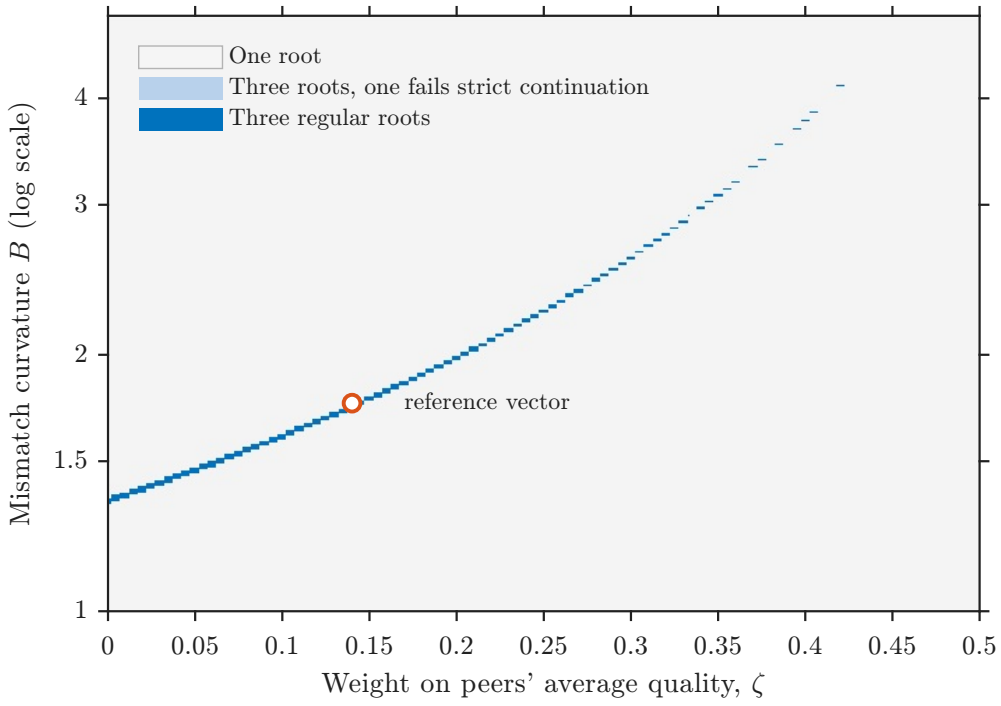


Figure 5: Number of stationary roots on K by peer weight ζ and mismatch curvature B, holding the other reference primitives (Table 1) fixed. Dark cells satisfy the strict-stopping condition of Definition 3 at each root; light cells satisfy it at a subset of the roots.

5 Quantification

5.1 Calibration

The replacement flow $\mathcal{J}$ is the share of buyers per unit of time that discard a durable and buy the frontier vintage in one product subline. With a common mass per discarded unit, discarded mass per buyer is proportional to $\mathcal{J}$. Physical depreciation occurs at the common rate δ. Economic obsolescence depends in addition on frontier growth and the peer benchmark: a unit reaches the replacement threshold after the replacement age $q^*/(\delta + g)$. Under a common physical depreciation rate, two countries discard the same product at different rates when innovation and peer comparisons differ.

Seller research expenditure relative to sales in a product subline is

$$\frac{\omega g^2}{(\kappa + \pi^S)\mathcal{J}}. \tag{32}$$

The research intensity (32) and the replacement flow are evaluated at each stationary equilibrium, with common δ, B, and ζ. Across stationary equilibria, both rise together, as electronic waste and R&D expenditure do across OECD countries.

Time is measured in years. The discount rate is $\rho = 0.08$. Attanasio et al. (2022) estimate that a car under regular maintenance loses 11.9 percent of its quality per year, so $e^{-\delta} = 0.881$ and $\delta = -\log 0.881 = 0.1267$. The normalization $A = 1$ fixes the service-payoff scale. The gross markup is $\eta = 1.61$, the 2016 aggregate markup in De Loecker et al. (2020), so the product-subline demand elasticity is $\sigma = 2.64$ and the unit production cost is $\kappa = (\kappa+\pi^S)/\eta$. The frontier price of a unit with the threshold quality $\underline{x}(t)$ is $(\kappa + \pi^S)\underline{x}(t)$, so $\chi/(\kappa + \pi^S)$ is the fixed replacement cost as a fraction of the value of the discarded durable. Berger and Vavra (2015) estimate the fraction of durable value lost at adjustment at 0.0525, which sets $\chi = 0.0525(\kappa + \pi^S)$.

Two observed moments discipline the fast-replacement equilibrium. The price index for durable goods in personal consumption expenditures falls relative to the overall index at 2.99 percent per year over 2007–2024 (Appendix A.5). Treating this decline as frontier quality improvement sets $g = 0.0299$ in H. In the 2017 National Household Travel Survey, 13.8 percent of household vehicles were acquired in the preceding year (U.S. Department of Transportation, Federal Highway Administration, 2017), which sets $\mathcal{J} = 0.1378$ in H, a replacement age of 7.3 years. Given these targets, the seller condition (26) determines ω and the buyer condition (25) determines the price per unit of frontier quality, $\kappa + \pi^S = 0.545$. This price divides into a unit production cost $\kappa = 0.339$ and a seller margin $\pi^S = 0.207$, and the fixed replacement cost is $\chi = 0.0286$.

The peer response in phone purchases disciplines the mismatch curvature B. In Bailey et al. (2022), a user has 328 friends on average, and one additional friend who buys a new phone after a random phone loss raises the user's probability of buying a new phone in the following week by 0.041 percentage points. In the model, the friend replaces a unit of average installed quality with a frontier unit, which raises Q by $(e^{q^*-Q}-1)/328$ and the peer benchmark by ζ times that amount. Holding the other peers' choices fixed, the buyer's replacement threshold rises, and the buyers between the old and the new threshold replace at once. Their share, the threshold shift divided by q^*, is the model counterpart of the one-week response. Given ζ, the one-week response determines B. Along this relation, three regular stationary equilibria with H as the fast-replacement equilibrium exist on a ζ grid of step 0.0025 for $\zeta \in [0.1375, 0.1425]$. The reference peer weight is $\zeta = 0.140$, with $B = 1.755$.

Table 1: Parameter values.

Parameter	Target or role	Value
External		
ρ	Annual discount rate	0.080
η	Gross markup, De Loecker et al. (2020)	1.610
δ	Annual car quality loss of 11.9 percent, Attanasio et al. (2022)	0.1267
$\chi/(\kappa+\pi^S)$	Fraction of durable value lost at adjustment, Berger and Vavra (2015)	0.0525
Internal		
ω	Relative durable price decline of 2.99 percent	7.3066
$\kappa+\pi^S$	Annual vehicle acquisition share of 13.78 percent	0.5452
B	One-week peer response of 0.041 percentage points	1.7549
ζ	Three-equilibrium range $[0.1375, 0.1425]$	0.140
Implied		
$\kappa = (\kappa+\pi^S)/\eta$	Unit production cost	0.3386
π^S	Seller margin per unit of frontier quality	0.2066
χ	Fixed replacement cost	0.0286
$\sigma = \eta/(\eta-1)$	Product-subline demand elasticity	2.639
Normalization		
A	Service-payoff scale	1.000

Notes: Table 5 reports the OECD moment targets for the cross-country aggregation. Unrounded values: $\delta = 0.13$, $\omega = 7.31$, $\kappa+\pi^S = 0.55$, $\kappa = 0.34$, $\pi^S = 0.21$, $\chi = 0.03$, $B = 1.75$.

Table 2 reports the calibration at the two endpoints of this range and at the reference peer weight. At each peer weight, B, ω, and $\kappa+\pi^S$ solve the one-week peer response, the seller condition, and the buyer condition. The fast-replacement equilibrium is the same across the three peer weights. As ζ rises, the slow and middle equilibria move toward each other and the price per unit of frontier quality rises.

Table 2: Calibration across peer weights.

				L		M		H	
ζ	B	ω	$\kappa+\pi^S$	g (%)	Age	g (%)	Age	g (%)	Age
0.1375	1.787	6.92	0.516	1.20	15.9	2.77	7.7	2.99	7.3
0.1400	1.755	7.31	0.545	1.31	14.7	2.35	8.8	2.99	7.3
0.1425	1.724	7.68	0.573	1.59	12.4	1.80	11.0	2.99	7.3

Notes: Age is the replacement age in years. At each ζ, B matches the one-week peer response of 0.041 percentage points, ω solves the seller condition and $\kappa+\pi^S$ solves the buyer condition at the fast-replacement targets, and $\chi = 0.0525(\kappa+\pi^S)$; the remaining parameters are those of Table 1. Welfare is highest in H at each ζ.

5.2 Stationary equilibria

Figure 6 plots the buyer and seller loci, and Table 3 reports frontier growth and replacement flows at their intersections.

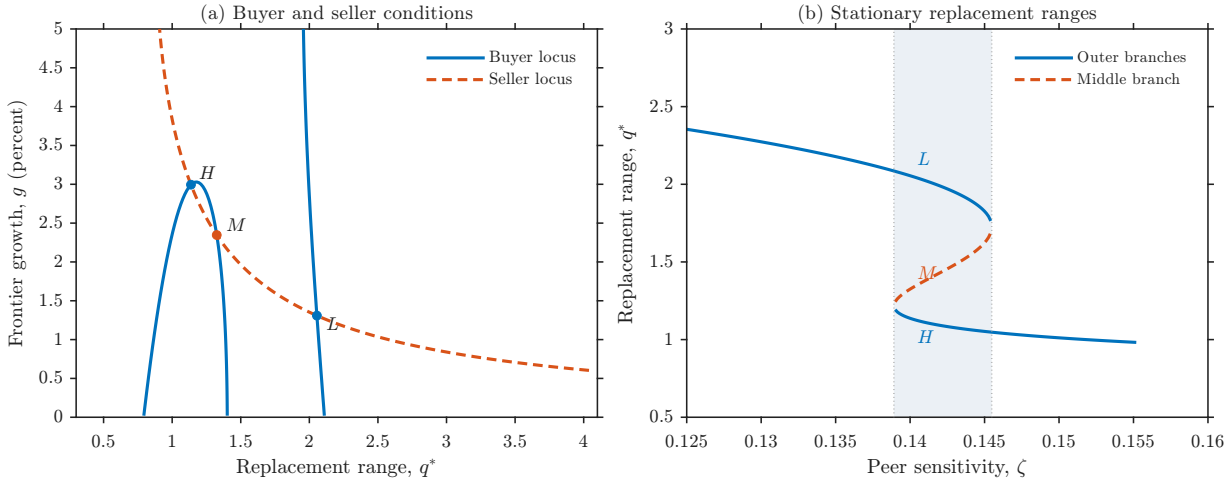


Figure 6: Equilibrium replacement and growth. Panel (a) plots buyer and seller loci in (q^*, g); intersections are regular BGP roots. Panel (b) plots replacement ranges as the peer weight ζ varies; the shaded region marks the computed three-root window.

Stationary equilibrium	g (%)	q^*	Replacement flow $\mathcal{J}$ (%)	Seller R&D / sales (%)
H: fast replacement	2.99	1.14	13.78	8.72
M: middle	2.35	1.32	11.33	6.51
L: slow replacement	1.31	2.06	6.80	3.38

Table 3: Stationary equilibria under the reference parameter vector of Table 1. Seller R&D intensity is (32).

The fast- and slow-replacement equilibria share the physical depreciation rate. The replacement flow in H is 2.0 times that in L, and seller R&D intensity is 2.6 times as large. The

replacement age is 14.7 years in L and 7.3 years in H. Section 5.3 converts the one-subline replacement flow into kilograms of electronic waste.

Let W be the discounted value of services less the mismatch loss and resource costs along a balanced-growth path, per unit of initial frontier quality. Table 4 reports its components. Welfare is highest in H and lowest in L. Services account for most of the difference: older installed durables in L deliver fewer services per unit of frontier quality.

Table 4: Welfare in the three stationary equilibria, per unit of frontier quality.

Stationary equilibrium	Services	Mismatch loss	Resource costs	W
L: slow replacement	0.424	0.245	0.025	2.31
M: middle	0.554	0.214	0.043	5.25
H: fast replacement	0.597	0.191	0.054	7.03

Notes: Services are $\mathcal{A}X/\bar{x}$. The mismatch loss is the cross-sectional average of $\mathcal{B}[\log x - M]^2$ divided by $\bar{x}$, with $\mathcal{B} = B\underline{x}$. Resource costs are production, the fixed replacement cost, and R&D, each divided by $\bar{x}$. Parameter vector of Table 1.

5.3 Cross-country calibration and model fit

Each country has two equally sized groups of product sublines with the parameter vector of Table 1. Each group contains a continuum of product sublines, so each seller is atomistic. Each group occupies the fast-replacement equilibrium with probability p_H and the slow-replacement equilibrium otherwise, independently across groups. A country therefore has zero, one, or two fast-equilibrium groups. Its replacement flow is the average flow across groups, its average installed quality is the average of $X/\bar{x}$ across groups, and its seller R&D intensity is total seller research expenditure divided by total seller sales. A common effective installed mass m converts the replacement flow into generated e-waste in kilograms per person.

I choose m and p_H to match the mean and standard deviation of generated e-waste across OECD countries. Table 5 reports the moments.

Figure 7 places the OECD observations and the three model equilibria for three common peer weights on a common e-waste scale. The horizontal variables are centered and divided by their own standard deviations. Panel (a) pairs OECD e-waste with GDP per capita and predicted e-waste with installed quality. Panel (b) pairs OECD e-waste with national R&D relative to GDP and predicted e-waste with seller R&D intensity. The other two curves vary the common peer weight and hold the e-waste mass and the fast-equilibrium probability fixed. The model predicts an upward pattern in both panels.

Table 5: Cross-country calibration and model moments.

Parameter	Value	Moment	OECD data	Model
m	173.95	Mean e-waste (kg per person)	18.34	18.34
p_H	0.536	E-waste standard deviation (kg per person)	4.28	4.28
		Lowest eight-country mean (kg per person)	11.90	11.83
		Highest eight-country mean (kg per person)	23.29	23.96
		Income–quality correlation	0.77	1.00
		Research correlation	0.44	0.97

Notes: The two top rows are fitted. Income–quality is e-waste/GDP per capita in the data and replacement/installed quality in the model. Research is e-waste/national R&D relative to GDP with replacement/seller R&D relative to sales.

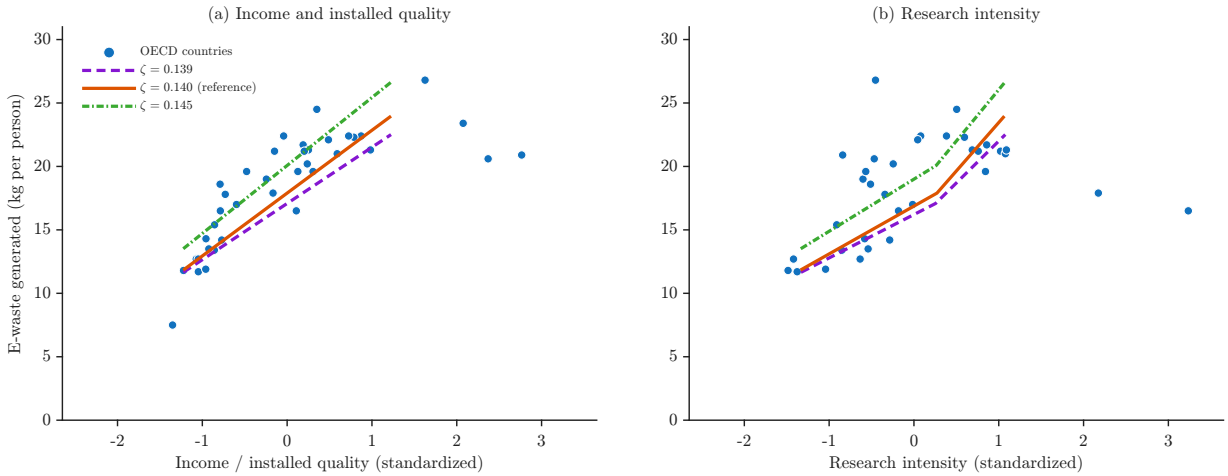


Figure 7: OECD observations and model equilibria across peer weights. Blue points are 2022 OECD countries. Each line connects three model equilibria, from zero to two groups in the fast-replacement equilibrium. The solid orange line uses the reference peer weight $\zeta = 0.140$; the dashed purple and dash-dotted green lines use $\zeta = 0.139$ and $\zeta = 0.145$. The effective mass and fast-equilibrium probability are common across the three lines. Panel (a) compares GDP per capita with installed quality. Panel (b) compares national R&D/GDP with seller R&D/sales. The parameter vector is common across product sublines.

Peer-weight sensitivity. Each row of Table 6 holds ζ fixed across the hypothetical countries and reports the slow- and fast-replacement equilibria. The common ζ varies across rows.

Common ζ	$\mathcal{J}_L$ (%)	$\mathcal{J}_H$ (%)	R&D/sales$_L$ (%)	R&D/sales$_H$ (%)
0.1390	6.70	12.94	3.31	7.91
0.1400	6.80	13.78	3.38	8.72
0.1425	7.13	14.70	3.57	9.70
0.1450	7.77	15.31	3.97	10.41

Table 6: Slow- and fast-replacement equilibria as the common peer weight varies. Other primitives, including physical depreciation and mismatch curvature, are fixed at Table 1. L and H are the slow- and fast-replacement stationary equilibria.

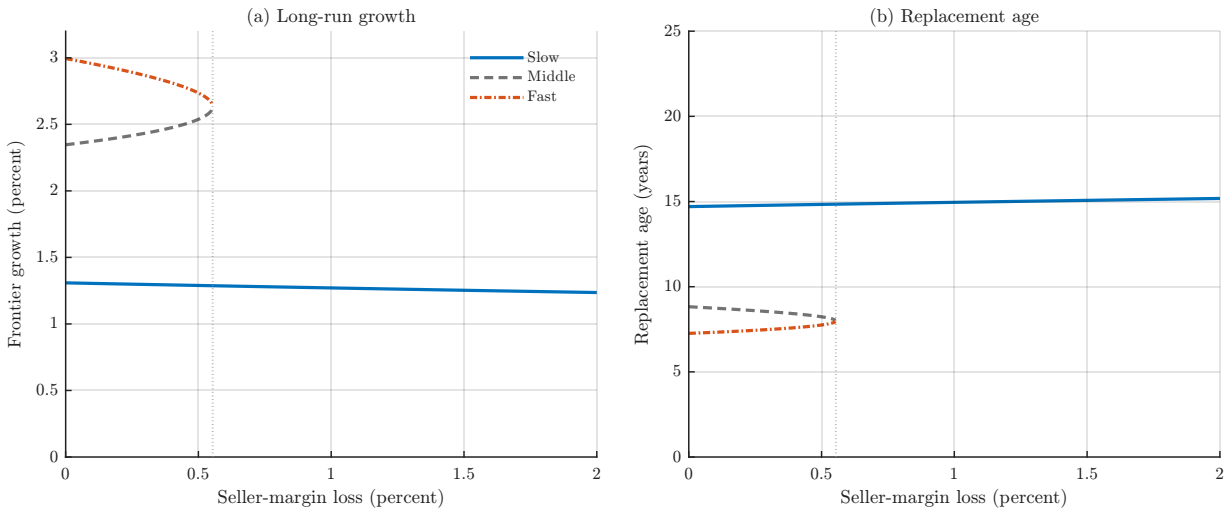


Figure 8: Stationary effects of a production-cost increase. The solid blue, dashed gray, and dash-dotted orange lines are the slow, middle, and fast stationary equilibria. Panel (a) reports frontier growth. Panel (b) reports replacement age. Each point solves the stationary buyer and seller conditions at the indicated seller-margin loss. The dotted vertical line marks the 0.55 percent loss at which the middle and fast equilibria meet.

6 Production Costs and Long-Run Growth

Memory prices have risen as demand from AI data centers has drawn supply toward servers. PC and smartphone producers face higher component costs (TrendForce, 2026). Industry forecasts connect the memory shortage to lower device shipments and longer replacement ages. Gartner (2026) projects 2026 PC and smartphone shipments to fall by 10.4 and 8.4 percent, and PC lifetimes to rise by 15 percent for business buyers and 20 percent for consumers by the end of 2026. Replacement flow and replacement age are the model counterparts of shipments and lifetimes. A persistent increase in production cost also lowers the seller's return to innovation and, through it, long-run frontier growth.

Consider an unexpected permanent increase in the unit production cost κ, together with a fall in the gross markup η.[1] Under (6), a higher κ at a fixed η raises the seller margin; the experiment lowers η so that π^S falls by 0.25, 0.5, or 10 percent of its initial value. The increase in κ is four times the fall in π^S, so one quarter of the cost increase lowers the seller margin and three quarters raises the buyer price. At the 10 percent loss, η falls from 1.61 to 1.44, and the implied demand elasticity σ rises from 2.6 to 3.3. The remaining parameters retain the values in Table 1.

Figure 8(a) plots the stationary annual growth rate against the seller-margin loss. The

[1]The directed-search model of Ceneri (2026a) determines waiting times for durable purchases in equilibrium, across quality submarkets. Embedding the cost increase in that model endogenizes the rise in waiting times that accompanies higher production costs.

Table 7: Stationary equilibria after a permanent production-cost increase.

Seller-margin loss	Equilibrium	Replacement flow (%)	Replacement age (years)	Frontier growth (%)	30-year shortfall (%)
0%	Fast	13.78	7.26	2.994	0.0
0.25%	Fast	13.44	7.44	2.891	3.1
0.5%	Fast	12.90	7.75	2.736	7.5
10%	Slow	6.03	16.57	1.025	44.6

Notes: The 30-year shortfall is the percent gap between frontier quality and the initial fast-replacement path after 30 years at the stationary growth rate.

dash-dotted orange line is the fast-replacement equilibrium. Its growth rate falls from 2.99 percent to 2.89 percent at the 0.25 percent loss and to 2.74 percent at the 0.5 percent loss. The dashed gray line is the middle equilibrium. Its growth rate rises from 2.35 percent and meets the fast-replacement equilibrium at the 0.55 percent loss. At larger losses, the slow-replacement equilibrium is the only stationary equilibrium. The solid blue line is the slow-replacement equilibrium. Its growth rate falls from 1.31 percent to 1.29 percent at the 0.5 percent loss and to 1.03 percent at the 10 percent loss.

Figure 8(b) converts replacement flow into replacement age. The orange fast-replacement line rises from 7.26 years to 7.44 years at the 0.25 percent loss and to 7.75 years at the 0.5 percent loss. The gray middle line falls from 8.8 years and meets the orange line. The blue slow-replacement line rises from 14.7 years to 16.6 years at the 10 percent loss. At the 10 percent loss, the replacement age of the stationary equilibrium rises from 7.3 to 16.6 years. Table 7 reports replacement flow, replacement age, and frontier growth at the three seller-margin losses.

The last column of Table 7 converts the stationary growth rates into frontier-quality shortfalls. The calculation applies the stationary growth rate from the shock date. After the 10 percent loss, the slow-replacement equilibrium is the only stationary equilibrium. Relative to the initial fast-replacement path, the frontier-quality shortfalls after 30 years are 3.1, 7.5, and 44.6 percent.

A temporary cost increase restores the initial fast-replacement growth rate at recovery. The slow-replacement equilibrium also exists after recovery, so the return to H is an equilibrium selection. A five-year cost increase leaves frontier quality 1.3 percent below the initial path after the 0.5 percent loss and 9.4 percent below it after the 10 percent loss. Growth returns to its initial rate at recovery, so the proportional frontier-quality gap remains fixed thereafter.

Figure 9 extends the experiment to a range of peer weights. The horizontal axis reports the unit production cost κ along the experiment, from a seller-margin gain of 5 percent to a

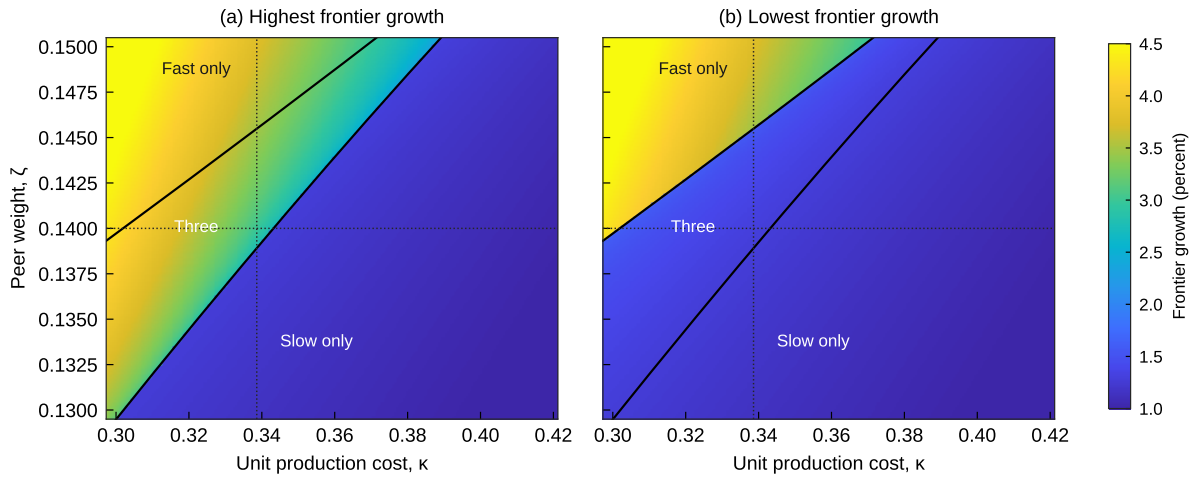


Figure 9: Highest and lowest stationary frontier growth across unit production costs and peer weights. The unit production cost κ and the seller margin π^S move together as in the production-cost experiment. Panel (a) reports the highest stationary frontier growth, and panel (b) reports the lowest. The black curves bound the region with three stationary equilibria, and the two panels coincide outside it. The dotted lines mark the reference parameter values in Table 1, which also gives the remaining parameters. Stationary roots are computed on $(0, \rho)$.

seller-margin loss of 10 percent. The vertical axis reports the peer weight ζ. Panel (a) reports the highest stationary frontier growth, and panel (b) reports the lowest. The two black curves bound the region with three stationary equilibria. Inside this region, panel (a) reports the fast-replacement equilibrium and panel (b) reports the slow-replacement equilibrium. Outside this region, the stationary equilibrium is unique and the two panels coincide. A higher peer weight moves both curves toward higher unit production costs. At $\zeta = 0.130$, the fast-replacement equilibrium exists for seller-margin gains above 4.4 percent. At $\zeta = 0.140$, a seller-margin loss above 0.55 percent removes the fast-replacement equilibrium. At $\zeta = 0.150$, the fast-replacement equilibrium is the unique stationary equilibrium up to a 3.6 percent loss and coexists with the slow-replacement equilibrium up to a 5.8 percent loss. At every peer weight in the figure, a seller-margin loss above 6.1 percent leaves the slow-replacement equilibrium as the unique stationary equilibrium.

The seller's market size links the cost increase to long-run innovation. A temporary cost increase produces a permanent frontier-quality level gap. A permanent cost increase produces a growth-rate gap and a widening frontier-quality gap. A seller-margin loss above 0.55 percent removes the fast-replacement equilibrium, and frontier growth falls to the slow-replacement rate.

7 Conclusion

Buyer replacement and seller innovation reinforce each other. A buyer replaces sooner when peers hold newer durables, and more purchases raise the seller's return to improving the next vintage. When durable services fall with the distance from the average quality of peers' durables, the same preferences, costs, and physical depreciation support a slow-replacement equilibrium and a fast-replacement equilibrium. In the slow-replacement equilibrium, buyers keep their durables longer and the seller innovates slowly. In the fast-replacement equilibrium, durables become obsolete sooner, purchases are more frequent, and innovation is faster. Welfare is highest in the fast-replacement equilibrium.

After peers move to newer products, the replacement incentive rises most for buyers holding the oldest vintages. Purchase dates, product vintages, and peer networks measure that response. A permanent rise in production costs lowers frontier growth, and a seller-margin loss above 0.55 percent removes the fast-replacement equilibrium.

References

Adlakha, S. and Johari, R. (2013). Mean Field Equilibrium in Dynamic Games with Strategic Complementarities. *Operations Research*, 61(4):971–989.

Aghion, P., Boppart, T., Peters, M., Schwartzman, M., and Zilibotti, F. (2025). A Theory of Endogenous Degrowth and Environmental Sustainability. Working Paper 33634, National Bureau of Economic Research.

Alvarez, F. E., Argente, D., Lippi, F., Méndez, E., and Van Patten, D. (2023a). Strategic Complementarities in a Dynamic Model of Technology Adoption: P2P Digital Payments. Working Paper 31280, National Bureau of Economic Research.

Alvarez, F. E., Lippi, F., and Souganidis, P. E. (2023b). Price Setting with Strategic Complementarities as a Mean Field Game. *Econometrica*, 91(6):2005–2039.

Attanasio, O., Larkin, K. P., Ravn, M. O., and Padula, M. (2022). (S)Cars and the Great Recession. *Econometrica*, 90(5):2319–2356.

Bailey, M., Johnston, D., Kuchler, T., Stroebel, J., and Wong, A. (2022). Peer Effects in Product Adoption. *American Economic Journal: Applied Economics*, 14(3):488–526.

Baldé, C. P., Kuehr, R., Yamamoto, T., McDonald, R., D'Angelo, E., Althaf, S., Bel, G., Deubzer, O., Fernandez-Cubillo, E., Forti, V., Gray, V., Herat, S., Honda, S., Iattoni, G.,

Khetriwal, D. S., Luda di Cortemiglia, V., Lobuntsova, Y., Nnorom, I., Pralat, N., and Wagner, M. (2024). The Global E-waste Monitor 2024. Technical report, International Telecommunication Union and United Nations Institute for Training and Research.

Benhabib, J., Perla, J., and Tonetti, C. (2021). Reconciling Models of Diffusion and Innovation: A Theory of the Productivity Distribution and Technology Frontier. *Econometrica*, 89(5):2261–2301.

Berger, D. and Vavra, J. (2015). Consumption Dynamics During Recessions. *Econometrica*, 83(1):101–154.

Bertolotti, F., Gavazza, A., and Lanteri, A. (2023). Dynamics of Expenditures on Durable Goods: The Role of New-Product Quality. *The Economic Journal*, 133(652):1641–1656.

Bertolotti, F. and Lanteri, A. (2024). Capital Replacement and Innovation Dynamics. CEPR Discussion Paper 18869, Centre for Economic Policy Research. Revised May 2026.

Bertolotti, F., Lanteri, A., and Villa, A. T. (2024). Investment-goods market power and capital accumulation. Working Paper 2024-13, Federal Reserve Bank of Chicago.

Caballero, R. J. (1993). Durable Goods: An Explanation for Their Slow Adjustment. *Journal of Political Economy*, 101(2):351–384.

Caplin, A. and Leahy, J. (2006). Equilibrium in a Durable Goods Market with Lumpy Adjustment. *Journal of Economic Theory*, 128(1):187–213.

Ceneri, N. (2026a). A Directed-Search Theory of Durable Goods Replacement. SSRN Working Paper No. 6621258. https://doi.org/10.2139/ssrn.6621258.

Ceneri, N. (2026b). Coordination in Technology Adoption: Networks, Innovation, and Growth. Working paper.

De Loecker, J., Eeckhout, J., and Unger, G. (2020). The rise of market power and the macroeconomic implications. *Quarterly Journal of Economics*, 135(2):561–644.

Eberly, J. C. (1994). Adjustment of Consumers' Durables Stocks: Evidence from Automobile Purchases. *Journal of Political Economy*, 102(3):403–436.

Farrell, J. and Saloner, G. (1985). Standardization, Compatibility, and Innovation. *The RAND Journal of Economics*, 16(1):70–83.

Gartner (2026). Surging Memory Costs Will Reduce Global PC and Smartphone Shipments in 2026. Gartner Newsroom.

Goettler, R. L. and Gordon, B. R. (2011). Does AMD Spur Intel to Innovate More? *Journal of Political Economy*, 119(6):1141–1200.

Greenwood, J., Hercowitz, Z., and Krusell, P. (1997). Long-Run Implications of Investment-Specific Technological Change. *American Economic Review*, 87(3):342–362.

Grossman, G. M. and Helpman, E. (1991). Quality Ladders and Product Cycles. *The Quarterly Journal of Economics*, 106(2):557–586.

Grossman, S. J. and Laroque, G. (1990). Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods. *Econometrica*, 58(1):25–51.

Katz, M. L. and Shapiro, C. (1985). Network Externalities, Competition, and Compatibility. *American Economic Review*, 75(3):424–440.

Lasry, J.-M. and Lions, P.-L. (2007). Mean Field Games. *Japanese Journal of Mathematics*, 2(1):229–260.

Perla, J. and Tonetti, C. (2014). Equilibrium Imitation and Growth. *Journal of Political Economy*, 122(1):52–76.

TrendForce (2026). AI Server Demand Continues to Support Memory Prices in 3Q26, but Gains Moderate as Consumer Demand Weakens and High Base Effects Take Hold. TrendForce Press Center.

U.S. Bureau of Economic Analysis (2025). Personal consumption expenditures: Durable goods (chain-type price index) and personal consumption expenditures (chain-type price index), series DDURRG3A086NBEA and DPCERG3A086NBEA. Retrieved from FRED, Federal Reserve Bank of St. Louis.

U.S. Department of Transportation, Federal Highway Administration (2017). 2017 national household travel survey. Public-use household and vehicle files.

A Numerical Implementation and Parameterization

A.1 Root and transition diagnostics

The reported roots satisfy $F_B = F_S = 0$, $q^* > 0$, and $0 < g < \rho$. Independent quadrature gives a maximum absolute buyer residual of 6.8×10^{-16} across the three roots in Table 3, and the seller residuals equal zero at reported precision. At L, M, and H, the finite-difference determinants of the stationary buyer–seller Jacobian are -1.400, 0.146, and -0.135. On a 20,001-point grid, continuation has positive value relative to immediate replacement throughout the interior at each root. The three roots therefore satisfy the conditions of Definition 3. A search over 60,000 growth rates in $(0, 0.075)$ finds three roots inside K at the reference vector and at the 0.25 and 0.5 percent seller-margin losses, and a single root, the slow-replacement equilibrium, at the 10, 15, 25, 80, and 90 percent losses. At the reference curvature, three regular roots exist for $\zeta \in [0.13891, 0.14548]$.

Table 8: Equilibrium transitions after a permanent production-cost increase, from the inherited fast-replacement stock to the post-shock fast-replacement equilibrium.

Seller-margin loss	g at impact (%)	Stationary g (%)	$\max_t \lvert\gamma_t - g\rvert$ (pp)	Buyer residual	Seller residual	30-year shortfall (%)
0.25%	2.882	2.891	0.010	2.5×10^{-5}	2.1×10^{-10}	3.04
0.5%	2.698	2.736	0.037	1.1×10^{-4}	2.5×10^{-10}	7.44

Notes: Horizon 30 years, 60 time steps, 600 age cells. The buyer residual is the maximum absolute value-matching residual in threshold units. The 30-year shortfall is $1 - \exp\{\int_0^{30} (\gamma_t - g_H)\, \mathrm{d}t\}$, with g_H the initial fast-replacement growth rate; the stationary calculation gives 3.05 and 7.47 percent. Buyer values in H are of order 1.

A.2 Nonlinear transition system

Let $h_t = \mathrm{d} \log \underline{x}_t / \mathrm{d}t$ denote threshold growth and $\gamma_t = \mathrm{d} \log \bar{x}_t / \mathrm{d}t$ denote frontier growth. The replacement range is $q_t^* = \log(\bar{x}_t / \underline{x}_t)$. On the fixed coordinate $y = q/q_t^* \in [0, 1]$, let $\tilde{\psi}(y, t)$ denote the installed-vintage density. Its transport equation is

$$\tilde{\psi}_t + \partial_y \left\{ -\frac{\delta + h_t + y(\gamma_t - h_t)}{q_t^*} \tilde{\psi} \right\} = 0. \tag{33}$$

Discarded durables return to frontier quality, with boundary flux $(\delta + \gamma_t)\tilde{\psi}(1, t) = (\delta + h_t)\tilde{\psi}(0, t)$. The replacement flow and average installed quality are

$$\mathcal{J}_t = \frac{\delta + h_t}{q_t^*} \tilde{\psi}(0, t), \qquad Q_t = \log\left(\int_0^1 e^{q_t^* y} \tilde{\psi}(y, t)\, \mathrm{d}y \right). \tag{34}$$

Let $w(y,t)$ denote the threshold-normalized buyer value on the fixed coordinate y. It satisfies

$$w_t = (\rho - h_t)w - u(q_t^* y, Q_t) + \frac{\delta + h_t + y(\gamma_t - h_t)}{q_t^*} w_y, \tag{35}$$

with smooth pasting $w_y(0,t) = 0$ and value matching

$$w(1,t) - w(0,t) = (\kappa + \pi^S)e^{q_t^*} + \chi. \tag{36}$$

The seller's shadow value and frontier growth satisfy

$$\gamma_t = \frac{s_t}{2\omega}, \qquad \dot{s}_t = (\rho - \gamma_t)s_t - \pi^S \mathcal{J}_t + \omega\gamma_t^2. \tag{37}$$

The replacement range evolves according to $\dot{q}_t^* = \gamma_t - h_t$. The installed-vintage distribution is inherited at the shock date. A jump in the replacement range places the inherited vintages on the new support, and buyers beyond the new threshold replace immediately. Characteristic renewal carries the distribution forward (Appendix B.9).

Along characteristics, the profile of the installed-vintage density is invariant. An inherited density that differs from the uniform stationary density therefore recurs in every replacement cycle, and the replacement flow $\mathcal{J}_t$ fluctuates around its stationary value at every later date. Convergence of a transition is measured by frontier growth, the buyer and seller residuals, and mass conservation.

The solution method takes the replacement range q_t^* on a time grid as the unknown. Given q_t^* and γ_t, a conservative renewal scheme propagates the inherited density and gives $\mathcal{J}_t$ and Q_t. A backward trapezoid recursion solves (37) for s_t, and hence γ_t, given $\mathcal{J}_t$; iteration on γ_t converges in a few passes. The buyer value follows from exact cohort paths and value matching (36). Nonlinear least squares sets the value-matching residuals to zero, with the post-shock stationary value as the terminal condition. Table 8 reports the transitions after the 0.25 and 0.5 percent losses.

A.3 Verification of the regular turning point

At the reference parameter vector, numerical computations verify Assumption 1 for the turning point at which the middle and fast-replacement equilibria appear, with the slow-replacement equilibrium as the background root. Let N_F be an interval of growth rates around the turning point g_F and N_B an interval around the background root g_B. Figure 10 plots the derivative sign change, the background sign crossing, and the residual away from those roots. The minima in Table 9 are computed on fine grids over each interval.

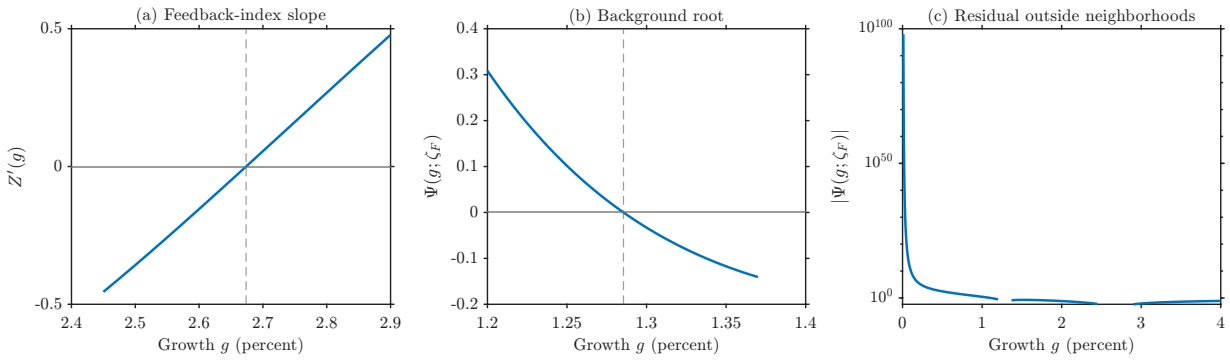


Figure 10: Numerical checks for the turning-point conditions at the reference vector. Panel (a) plots $Z'(g)$ on N_F. Panel (b) plots the residual on N_B at ζ_F. Panel (c) plots $|\Psi(g;\zeta_F)|$ on K outside N_F and N_B.

Table 9: Numerical diagnostics for Assumption 1 at the reference vector. The points n_- and n_+ are the endpoints of N_F; b_- and b_+ are the endpoints of N_B.

Object	Value	Condition
Interval N_F	$[0.02450, 0.02900]$	turning-point neighborhood
Turning-point growth rate g_F	0.026736	$Z'(g_F) = 0$
Turning-point peer weight $\zeta_F = Z(g_F)$	0.138914	$\zeta_F \in (0,1)$
$Z'(n_-), Z'(n_+)$	$-0.455,\ 0.478$	derivative sign change
Minimum grid value of Z'' on N_F	190.532	nondegenerate turning point
Minimum grid value of Z on N_F	0.138914	turning point is a minimum
Background interval N_B	$[0.01200, 0.01370]$	background neighborhood
Background root g_B	0.012855	simple root
$\Psi(b_-), \Psi(b_+)$ at ζ_F	$0.308,\ -0.141$	sign crossing
Minimum grid value of $\lvert\Psi_g\rvert$ on N_B	1.042e+02	simple root
Compact interval K	$[0.00010, 0.04000]$	root count domain
Minimum grid value of $\lvert\Psi\rvert$ off N_F, N_B	4.611e−03	isolation

A.4 Construction of the cross-country facts

The e-waste observations are 2022 generated kilograms per person from Table A2.4 of Baldé et al. (2024). Generated waste is discarded electrical and electronic equipment before collection, reuse, treatment, or export. China is shown as a labeled comparison. The OECD observations determine the fitted lines and correlations. The country income variable is real GDP per capita in constant 2015 U.S. dollars (World Development Indicators, NY.GDP.PCAP.KD); the research variable is R&D expenditure as a percentage of GDP (GB.XPD.RSDV.GD.ZS). Both are measured in 2022.

In the two-group calibration, each group occupies the fast-replacement equilibrium with probability p_H. A common effective installed mass m maps the average replacement flow

into e-waste in kilograms per person. The mean and variance are

$$\mathbb{E}[Y] = m\Big[\mathcal{J}_L + p_H(\mathcal{J}_H - \mathcal{J}_L)\Big],$$
$$\mathrm{Var}(Y) = \frac{m^2}{2}(\mathcal{J}_H - \mathcal{J}_L)^2 p_H(1 - p_H).$$

The OECD mean is 18.339 kilograms per person and its population standard deviation is 4.278. These moments give $p_H = 0.53640$ and $m = 173.95$ kilograms per person, and also $p_H = 0.17402$ and $m = 228.80$. The observed e-waste distribution has negative skewness, and the high-p_H solution has the same sign.

A.5 Parameter calibration

The parameters in Table 1 yield the three stationary equilibria in Section 5. Multiplying A, B, κ, π^S, χ, and ω by a common positive constant leaves the stationary roots (q^*, g) unchanged, so $A = 1$ sets the units of the remaining payoff and cost parameters.

The relative price decline of durable goods uses two annual chain-type price indexes from the U.S. Bureau of Economic Analysis, retrieved from FRED: personal consumption expenditures on durable goods (DDURRG3A086NBEA) and total personal consumption expenditures (DPCERG3A086NBEA), over 2007–2024.

At $B = 0$, the mismatch payoff is zero, and replacement, durable pricing, and seller R&D remain. At $\zeta = 0$, the peer benchmark equals the quality at which peers replace, and Figure 5 shows coexistence there. A positive ζ adds the average-quality channel of Proposition 2.

B Derivations and Proofs

B.1 Normalization

Lemma 1. *Let*

$$q \equiv \log\left(\frac{x}{\underline{x}(t)}\right), \qquad q^*(t) \equiv \log\left(\frac{\bar{x}(t)}{\underline{x}(t)}\right), \qquad \Delta(t) = \delta + g(t).$$

Let $\psi(q,t)\,\mathrm{d}q = f(x,t)\,\mathrm{d}x$ *and*

$$Q(t) = \log\left(\int_0^{q^*(t)} e^q \psi(q,t)\,\mathrm{d}q\right).$$

Define $A(t) \equiv \mathcal{A}(t)$ and $B(t) \equiv \mathcal{B}(t)/\underline{x}(t)$, and define the threshold-normalized value $z(q,t) = V(\underline{x}(t)e^q, t)/\underline{x}(t)$ and the centered value

$$v(q,t) = z(q,t) - c(t), \qquad \dot{c}(t) = (\rho - g(t))c(t) - A(t).$$

On a BGP with $A(t) = A$ and $B(t) = B$, $c = A/(\rho - g)$. Value matching and smooth pasting hold for v as for z, and v satisfies the BGP HJB (21).

Proof. Since $q_x = 1/x$ and $q_t|_x = -g(t)$, with $V = \underline{x}z$,

$$V_x = \frac{\underline{x}}{x} z_q, \qquad V_t|_x = \underline{x}[z_t - g z_q + g z].$$

Dividing the levels HJB by $\underline{x}$ gives

$$(\rho - g)z = A(t)e^q - B(t)(q - \zeta Q)^2 - (\delta + g)z_q + z_t.$$

Substituting $z = v + c$ and choosing $\dot{c} = (\rho - g)c - A(t)$ gives

$$(\rho - g)v = A(t)(e^q - 1) - B(t)(q - \zeta Q)^2 - (\delta + g)v_q + v_t.$$

On a BGP, $v_t = 0$, which gives (21). The centering subtracts the same $c(t)$ in every state, so value differences and the derivative $v_q = z_q$ equal those of z. ■

B.2 Proof of Proposition 1

Proof. On a BGP, the stationary KFE is

$$-\Delta\psi'(q) = 0, \qquad q \in (0, q^*),$$

with renewal flux $\mathcal{J} = \Delta\psi(0) = \Delta\psi(q^*)$ and mass $\int_0^{q^*} \psi(q)\,\mathrm{d}q = 1$. Hence $\psi(q) = 1/q^*$ and $\mathcal{J} = (\delta + g)/q^*$. The expression for $Q(q^*)$ follows from

$$\frac{X(t)}{\underline{x}(t)} = \int_0^{q^*} e^q \frac{1}{q^*}\,\mathrm{d}q = \frac{e^{q^*} - 1}{q^*}.$$

Differentiating $Q(q^*)$ gives Q' in (24). ■

B.3 Integrated buyer residual

Throughout this appendix v denotes the centered normalized value from Lemma 1. Rewrite the BGP HJB as

$$\Delta v'(q) + Rv(q) = A(e^q - 1) - B(q - \mu)^2, \qquad R = \rho - g, \quad \mu = \zeta Q(q^*).$$

Let $\beta = R/\Delta$ and $E = e^{-\beta q^*}$. Smooth pasting gives $Rv(0) = -B\mu^2$. Integrating the buyer differential equation from 0 to q^* and eliminating $v(0)$ yields

$$v(q^*) - v(0) = \frac{A}{\Delta}\int_0^{q^*} e^{-\beta(q^*-s)}(e^s - 1)\,\mathrm{d}s + B\left[\frac{(1-E)\mu^2}{R} - \frac{1}{\Delta}\int_0^{q^*} e^{-\beta a}(q^* - a - \mu)^2\,\mathrm{d}a\right].$$

Define

$$\mathcal{I}_e(q^*, \beta) = \frac{e^{q^*} - E}{1+\beta} - \frac{1-E}{\beta},$$

$$\mathcal{I}_2(q^*, \mu, \beta) = \frac{(q^* - \mu)^2}{\beta} - \frac{2(q^* - \mu)}{\beta^2} + \frac{2}{\beta^3} - E\left(\frac{\mu^2}{\beta} + \frac{2\mu}{\beta^2} + \frac{2}{\beta^3}\right),$$

and

$$\Xi(q^*, \mu; \Delta, R, \beta) = \frac{(1-E)\mu^2}{R} - \frac{\mathcal{I}_2(q^*, \mu, \beta)}{\Delta}.$$

Then

$$F_B(q^*, g; \Theta, \zeta) = A\frac{\mathcal{I}_e(q^*, \beta)}{\Delta} + B\Xi(q^*, \mu; \Delta, R, \beta) - (\kappa + \pi^S)e^{q^*} - \chi. \tag{38}$$

Expanding $\mathcal{I}_2$ in powers of μ, the μ^2 terms in Ξ cancel because $R = \beta\Delta$. Hence Ξ is affine in μ, with coefficient Ξ_1 in (28). Thus $\Psi(g; \Theta, \zeta) = F_B(q_S(g), g; \Theta, \zeta)$ is affine in ζ.

B.4 Proof of Proposition 2

Proof. The peer benchmark enters F_B through $\mu = \zeta Q$. Differentiating (38) gives

$$\frac{\partial F_B}{\partial Q} = B\zeta\Xi_1(q^*, \Delta, \beta).$$

Let $\ell = \beta q^* > 0$. Then

$$\frac{q^*}{\beta} - \frac{1 - e^{-\beta q^*}}{\beta^2} = \frac{\ell - (1 - e^{-\ell})}{\beta^2} > 0,$$

because $\ell > 1 - e^{-\ell}$ for $\ell > 0$. Hence $\Xi_1 > 0$. ∎

B.5 Proof of Proposition 3

Proof. For $\pi^S > 0$, $\omega > 0$, and $g \in (0, \rho)$, the seller residual (26) is equivalent to the positive seller curve (27). A regular BGP therefore satisfies $q = q_S(g; \Theta)$ and $\Psi(g; \Theta, \zeta) = 0$. A regular zero of Ψ with $q = q_S(g; \Theta)$ satisfies both stationary equations and is a regular BGP by Definition 3.

The buyer residual is affine in ζ, with

$$S_1(g; \Theta) = B\Xi_1(q_S(g), \Delta(g), \beta(g))Q(q_S(g)) > 0,$$

because $B > 0$, $Q(q_S(g)) > 0$ for $q_S(g) > 0$, and $\Xi_1 > 0$. Therefore $\Psi = 0$ is equivalent to $Z(g; \Theta) = \zeta$. ■

B.6 Proof of Corollary 1

Proof. For $B = 0$, the mismatch component of the buyer residual is zero, so Ψ and $\mathcal{E}_K(\zeta)$ are constant in ζ. For $B > 0$ and strictly monotone Z, the equation $Z(g; \Theta) = \zeta$ has at most one solution. Hence $\mathcal{E}_K(\zeta)$ contains at most one regular BGP. ■

B.7 Analytical derivatives

Write $E = e^{-\beta q}$ and $H_0 = 1 - E$. Let

$$\mathcal{D}_0(q, \beta) = \frac{q^2}{\beta} - \frac{2q}{\beta^2} + \frac{2H_0}{\beta^3}, \qquad \mathcal{D}_1(q, \beta) = \frac{q}{\beta} - \frac{H_0}{\beta^2}.$$

Then

$$F_B(q, g; \Theta, \zeta) = \frac{A\mathcal{I}_e(q, \beta) - B\mathcal{D}_0(q, \beta) + 2B\zeta Q(q)\mathcal{D}_1(q, \beta)}{\Delta} - (\kappa + \pi^S)e^q - \chi.$$

The first row of the stationary Jacobian is $(F_{B,q}, F_{B,g})$, obtained by differentiating the display above. The second row is

$$F_{S,q} = -\pi^S \frac{\delta + g}{q^2}, \qquad F_{S,g} = \frac{\pi^S}{q} - 2\omega(\rho - g).$$

Thus

$$J(q, g) = \begin{bmatrix} F_{B,q} & F_{B,g} \\ -\pi^S(\delta + g)/q^2 & \pi^S/q - 2\omega(\rho - g) \end{bmatrix}. \tag{39}$$

At a regular turning point, $\det J = 0$.

B.8 Proof of Theorem 1

Proof. At a root, $\Psi(g;\Theta,\zeta) = S_1(g;\Theta)[\zeta - Z(g;\Theta)]$. Therefore

$$\Psi_g(g;\Theta,\zeta) = -S_1(g;\Theta)Z'(g;\Theta)$$

at any root. At (g_F,ζ_F), $\Psi = 0$, $\Psi_g = 0$, and

$$\Psi_{gg}(g_F;\Theta,\zeta_F) = -S_1(g_F;\Theta)Z''(g_F;\Theta) \neq 0, \qquad \Psi_\zeta(g_F;\Theta,\zeta_F) = S_1(g_F;\Theta) > 0.$$

Thus (g_F,ζ_F) is a regular turning point of the BGP consistency equation. Taylor expansion gives

$$0 = S_1(g_F)(\zeta - \zeta_F) - \frac{1}{2}S_1(g_F)Z''(g_F)(g - g_F)^2 + o(|\zeta - \zeta_F| + |g - g_F|^2),$$

which yields the square-root expansion and the side condition $(\zeta - \zeta_F)/Z''(g_F) > 0$.

The background root g_B is simple because $\Psi_g(g_B;\Theta,\zeta_F) \neq 0$. The implicit-function theorem gives a unique nearby C^1 continuation $g_B(\zeta)$. By Assumption 1, compactness, and uniform continuity, all nearby roots on K continue from g_B or the turning point. The root count on K is one on the one-root side and three on the three-root side.

Differentiating the seller residual along the seller curve gives $q_S'(g) = -F_{S,g}/F_{S,q}$. Hence

$$\Psi_g = F_{B,q}q_S'(g) + F_{B,g} = \frac{F_{B,g}F_{S,q} - F_{B,q}F_{S,g}}{F_{S,q}} = -\frac{\det J}{F_{S,q}}.$$

Since $F_{S,q} \neq 0$, $\det J = 0$ at the turning point and $\det J \neq 0$ at nearby simple roots. ∎

B.9 Characteristic vintage accounting

Let $a(t) = \int_0^t (\delta + h_s)\,\mathrm{d}s$ and let $\varphi_0(q_0)$ denote the initial density of log quality q_0. On the moving support $q \in [0, q_t^*]$, unreplaced units satisfy $\dot{q} = -(\delta + h_t)$. Assume that along the path $q_t^* \geq \underline{q} > 0$ for a constant $\underline{q}$, that the boundary fluxes are nonnegative, and that the crossing speed $\delta + h_t$ is positive. A cohort born at time τ has relative log quality

$$q(t;\tau) = q_\tau^* - [a(t) - a(\tau)]$$

and remains in use while $q_\tau^* > a(t) - a(\tau)$. Conditional on $\{q_t^*, h_t, \gamma_t\}$, average quality is

$$e^{Q_t} = \int_{q_0 > a(t)} e^{q_0 - a(t)}\varphi_0(q_0)\,\mathrm{d}q_0 + \int_0^t \mathbf{1}\{q_\tau^* > a(t) - a(\tau)\}e^{q_\tau^* - [a(t)-a(\tau)]}\mathcal{J}_\tau\,\mathrm{d}\tau.$$

Mass preservation gives

$$1 = \int_{q_0 > a(t)} \varphi_0(q_0)\,\mathrm{d}q_0 + \int_0^t \mathbf{1}\{q_\tau^* > a(t) - a(\tau)\}\mathcal{J}_\tau\,\mathrm{d}\tau.$$

With a positive crossing speed, the renewal boundary is defined almost everywhere. Differentiating the mass equation at continuity points determines the unique boundary flux $\mathcal{J}_t$. Substitution into the quality equation gives Q_t. The renewal equation preserves mass. A bounded initial density and a continuous replacement range give an absolutely continuous distribution at later dates. An impact change in the replacement range places a mass of buyers at frontier quality, and characteristic renewal carries that cohort forward as a point mass.